

# High and low temperature behavior of a quantum group fermion gas

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## Abstract

We consider the simplest  $SU_q(2)$  invariant fermionic hamiltonian and calculate the low and high temperature behavior for the two distinct cases  $q > 1$  and  $q < 1$ . For low temperatures we find that entropy values for the Fermi case are an upper bound for those corresponding to  $q \neq 1$ . At high temperatures we find that the sign of the second virial coefficient depends on  $q$ , and vanishes at  $q = 1.96$ . An important consequence of this fact is that the parameter  $q$  connects the fermionic and bosonic regions, showing therefore that  $SU_q(2)$  fermions exhibit fractional statistics in three spatial dimensions.

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The relevance that quantum groups and quantum algebras [1] may have outside the scope of integrable models has been in the last few years a subject of much interest for both physicists and mathematicians. The motivation of introducing quantum group symmetries in either space-time or as internal degrees of freedom addressed new interesting approaches which could give a new insight on the diverse roles they may play in physics. Some of these approaches led to the formulation of noncommutative geometry [2, 3, 4], and numerous studies in quantum mechanics [5], field theory [6], molecular and nuclear physics [7]. The main objective in this article is to investigate the thermodynamics of a gas with constituents satisfying an algebra covariant under the group  $SU_q(N)$ . The thermodynamics of  $q$ -deformed systems [8] and their possible relevance to anyon statistics [9] has been studied by several authors exclusively in the context of the so called quons [10]. In our case, we consider as our starting point a very simple quantum group invariant hamiltonian. As the most natural choice we consider a free  $SU_q(2)$  invariant hamiltonian in terms of operators generating a  $SU_q(2)$  covariant algebra, which in the  $q \rightarrow 1$  limit will become a fermionic gas. We study this model for low (high density) and high (low density) temperatures. At low  $T$  we find that the entropy function for quantum group gases,  $q \neq 1$ , lies below the graph of the fermionic gas entropy. At a given temperature the lowest entropy values corresponds to the extreme case of  $q \rightarrow 0$ . At high  $T$  we find that the second virial coefficient as a function of the parameter  $q$  interpolates continuously from fermionic to bosonic behavior, and it vanishes at  $q = 1.96$ .

The  $SU_q(2)$  covariant algebra generated by the quantum group fermions  $\Psi_i$ ,  $i = 1, 2$ , is given by the following relations

$$\{\Psi_2, \overline{\Psi}_2\} = 1 \quad (1)$$

$$\{\Psi_1, \overline{\Psi}_1\} = 1 - (1 - q^{-2})\overline{\Psi}_2\Psi_2 \quad (2)$$

$$\Psi_1\Psi_2 = -q\Psi_2\Psi_1 \quad (3)$$

$$\overline{\Psi}_1\Psi_2 = -q\Psi_2\overline{\Psi}_1 \quad (4)$$

$$\{\Psi_1, \Psi_1\} = 0 = \{\Psi_2, \Psi_2\}, \quad (5)$$

which for  $q = 1$  become a  $SU(2)$  covariant fermionic algebra. The covariance is simply checked by applying the linear transformation  $\Psi'_i = \sum_{j=1}^2 T_{ij}\Psi_j$ , where the matrix  $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is the two dimensional representation of the quantum group  $SU_q(2)$  [12] and the matrix coefficients  $(a, b, c, d)$  generate the algebra

$$\begin{aligned} ab &= q^{-1}ba \quad , \quad ac = q^{-1}ca \\ bc &= cb \quad , \quad dc = qcd \\ db &= qbd \quad , \quad da - ad = (q - q^{-1})bc \\ \det_q T &\equiv ad - q^{-1}bc = 1. \end{aligned} \quad (6)$$

Requiring  $T$  to be unitary leads to the adjoint matrix  $\overline{T}$  given by

$$\overline{T} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}, \quad (7)$$

where the parameter  $q$  must be a real number. Hereafter, we will consider  $0 \leq q < \infty$ . In general,  $SU_q(N)$  covariant fermionic algebras can be written in terms of the  $R$ -matrix of  $\hat{A}_{N-1}^q$ , as given in Reference [13].

There is a clear distinction between the algebra in Equations (1)-(5) with the  $q$ -boson algebra. Quons  $a_i$  obey the relations

$$a_i a_j^\dagger - q a_j^\dagger a_i = \delta_{i,j}, \quad (8)$$

which has the boson and fermion limits for  $q = 1$  and  $q = -1$  respectively. No specific relation between annihilation (or creation) operators is known for  $q^2 \neq 1$  and Equation (8) is not covariant under  $SU_q(N)$ .

A representation of  $SU_q(N)$ -fermions in terms of fermionic operators  $\psi_i$  and  $\psi_j^\dagger$  is given by the relations

$$\Psi_m = \psi_m \prod_{l=m+1}^N (1 + (q^{-1} - 1)M_l), \quad (9)$$

$$\overline{\Psi}_m = \psi_m^\dagger \prod_{l=m+1}^N (1 + (q^{-1} - 1)M_l), \quad (10)$$

where  $M_l = \psi_l^\dagger \psi_l$  and  $\{\psi_i, \psi_j^\dagger\} = \delta_{ij}$ . In what follows we consider the simplest  $SU_q(2)$  invariant hamiltonian, which is

$$\mathcal{H} = \sum_{\kappa} \varepsilon_{\kappa} (\overline{\Psi}_{\kappa,1} \Psi_{\kappa,1} + \overline{\Psi}_{\kappa,2} \Psi_{\kappa,2}), \quad (11)$$

with the quantum group fields satisfying  $\{\overline{\Psi}_{\kappa,i}, \Psi_{\kappa',j}\} = 0$  for  $\kappa \neq \kappa'$ . Thus, in terms of the fermionic fields  $\psi_i$ , Equation (11) becomes the interacting hamiltonian

$$\mathcal{H} = \sum_{\kappa} \varepsilon_{\kappa} (M_{\kappa,1} + M_{\kappa,2} + (q^{-2} - 1)M_{\kappa,1}M_{\kappa,2}), \quad (12)$$

where  $M_{\kappa,i} = \overline{\psi}_{\kappa,i} \psi_{\kappa,i}$  is the fermionic number operator . The grand partition function  $\mathcal{Z}$  is

$$\mathcal{Z} = \prod_{\kappa} (1 + 2e^{-\beta(\varepsilon_{\kappa} - \mu)} + e^{-\beta(\varepsilon_{\kappa}(q^{-2} + 1) - 2\mu)}), \quad (13)$$

which at  $q = 1$  becomes the partition function of a fermion gas with two degrees of freedom. The average number of particles  $\langle M \rangle$  becomes then a function of the parameter  $q$  according to the equation

$$\langle M \rangle = \langle m_1 \rangle + \langle m_2 \rangle$$

$$= 2V \int \frac{e^{\beta(\mu - \frac{p^2}{2m})}(1 + e^{\beta(\mu - \frac{q^{-2}p^2}{2m})})d^3p}{(2\pi\hbar)^3(1 + 2e^{-\beta(\frac{p^2}{2m}-\mu)} + e^{-\beta(\frac{p^2}{2m}(q^{-2}+1)-2\mu)})}. \quad (14)$$

### *Low temperature and high density gas*

In a previous work [13] we studied Equation (14) for the extreme cases  $q \gg 1$  and  $q \ll 1$ . It was shown that for low temperatures the chemical potential  $\mu(T)$  for  $q \gg 1$  is almost identical to the Fermi case while that for  $q \ll 1$  this function has in addition a linear temperature dependent term which vanishes for  $q = 1$  and  $q \gg 1$ . In this section we calculate the function  $\langle M \rangle$  for the two cases  $q > 1$  and  $q < 1$ . In addition, we calculate the internal energy  $U$ , heat capacity  $C_v$  and entropy  $S$  and compare them with the fermionic,  $q = 1$ , case.

#### a) $q > 1$

A simple way to calculate Equation (14) is by considering that at  $T = 0$  the average number of particles is given by  $\langle M \rangle = \frac{2}{3(q^{-2}+1)^{3/2}}\lambda(2\mu_0)^{3/2}$ . Thus, we split this equation in the following integrals

$$\begin{aligned} \langle M \rangle &= \lambda \int_0^{2\mu/(q^{-2}+1)} \varepsilon^{1/2} d\varepsilon \\ &- \lambda \left( \int_0^\mu + \int_\mu^{2\mu/(q^{-2}+1)} \right) \varepsilon^{1/2} \frac{1 + e^{\beta(\varepsilon-\mu)}}{f(\varepsilon, \mu, q)} d\varepsilon \\ &+ \lambda \left( \int_{2\mu/(q^{-2}+1)}^{q^2\mu} + \int_{q^2\mu}^\infty \right) \varepsilon^{1/2} \frac{1 + e^{-\beta(q^{-2}\varepsilon-\mu)}}{f(\varepsilon, \mu, q)} d\varepsilon, \end{aligned} \quad (15)$$

where  $f(\varepsilon, \mu, q) = e^{\beta(\varepsilon-\mu)} + 2 + e^{-\beta(q^{-2}\varepsilon-\mu)}$  and  $\lambda = \frac{4\pi V(2m)^{3/2}}{(2\pi\hbar)^3}$ . The second and fifth integrations vanish as  $\beta\mu \rightarrow \infty$ , and the remaining terms lead to the result

$$\langle M \rangle \approx \frac{2}{3(q^{-2}+1)^{3/2}}\lambda(2\mu)^{3/2} + \frac{0.64\lambda}{(q^{-2}+1)^{3/2}\beta^2\sqrt{2\mu}}. \quad (16)$$

Equation (16) is very similar to the Fermi case in the context that it does not contain a linear term in  $T$ . The internal energy  $U$  is calculated from the grand potential  $\Omega = -\frac{1}{\beta} \ln \mathcal{Z}$

$$\begin{aligned} U &= \left( \frac{\partial \beta \Omega}{\partial \beta} + \mu M \right) \\ &= V \int \frac{p^2}{2m} \frac{(2 + (q^{-2} + 1)e^{\beta(\mu - \frac{q^{-2}p^2}{2m})})d^3p}{(2\pi\hbar)^3 f(\varepsilon, \mu, q)}, \end{aligned} \quad (17)$$

such that following the same procedure as in Equation (15) we obtain for the internal energy

$$U \approx \lambda \left[ \left( \frac{2}{q^{-2} + 1} \right)^{3/2} \frac{2}{5} \mu_0^{5/2} + \frac{0.64\sqrt{2\mu_0}}{(q^{-2} + 1)^{3/2}\beta^2} \right], \quad (18)$$

and the entropy

$$S \approx \lambda \frac{1.28\sqrt{2\mu_0}k^2T}{(q^{-2} + 1)^{3/2}}. \quad (19)$$

### b) $q < 1$

For  $q < 1$  the function  $\langle M \rangle$  was calculated in Reference [13]. We use this result as a starting point and then calculate some thermodynamic functions. The average number of particles is given by the equation

$$\langle M \rangle \approx \frac{\lambda\mu^{3/2}(1+q^3)}{3} + \frac{\lambda\sqrt{\mu}(1-q^3)\ln 3}{2\beta} + \frac{0.54\lambda(1+q^3)}{2\sqrt{\mu}\beta^2}, \quad (20)$$

which contains a linear temperature term that vanishes for  $q = 1$ . Reverting Equation (20) leads to the chemical potential function

$$\mu \approx \mu_0 \left[ 1 - \frac{(1-q^3)\ln 3}{(1+q^3)\mu_0} kT - \left( 0.54 - \frac{(1-q^3)^2 \ln^2 3}{4(1+q^3)^2} \right) \left( \frac{kT}{\mu_0} \right)^2 \right]. \quad (21)$$

The internal energy  $U$  is simply calculated from Equation (17) by splitting this integral in the intervals  $[0, q^2\mu]$ ,  $[q^2\mu, \mu]$  and  $[\mu, \infty]$ . The internal energy in terms of the chemical potential is given by the equation

$$U \approx \frac{\lambda}{2} \left[ \frac{2}{5}(q^3 + 1)\mu^{5/2} + \frac{\ln 3}{\beta}(1 - q^3)\mu^{3/2} + \frac{3}{\beta^2}0.54(1 + q^3)\sqrt{\mu} \right], \quad (22)$$

such that after replacement of Equation (21) into this equation gives for the internal energy

$$U \approx \frac{\lambda}{2} \left[ \frac{2}{5}(q^3 + 1)\mu_0^{5/2} + \left( 1.08(q^3 + 1) - \frac{(1 - q^3)^2}{2(1 + q^3)} \ln^2 3 \right) \frac{\mu_0^{1/2}}{\beta^2} \right]. \quad (23)$$

The linear term in  $T$  has canceled out and the heat capacity

$$\begin{aligned} C_v &= \left( \frac{\partial U}{\partial T} \right)_V \\ C_v &\approx \lambda k^2 \sqrt{\mu_0} T \left[ 1.08(q^3 + 1) - \frac{(1 - q^3)^2}{2(1 + q^3)} \ln^2 3 \right], \end{aligned} \quad (24)$$

vanishes at  $T = 0$  in accordance with the third law. There is no solution with  $q \in \mathbf{R}$  such that  $C_v = 0$ , and there is only one real solution for  $q < 1$  at a given temperature for each value of  $C_v$ .

Figure 1 shows the function  $s = S/\lambda k^2 \sqrt{\mu_0}$  as a function of  $T$  for different values of the parameter  $q$ , where the entropy  $S = \int_0^T \frac{C_v}{T'} dT'$ . From Figure 1 we see that the entropy is maximum for the Fermi case, and there are systems with  $q > 1$  sharing the entropy function with those with  $q < 1$ . Specifically, for every value  $q_1$  such that  $0.33 \leq q_1 < 0.91$  there is a solution  $q_2 > 1$  with the same entropy function. For other values of  $q_1$  there is no value  $q_2 > 1$  which would give the same entropy at a given  $T$ . Therefore, for low temperatures entropy functions given by hamiltonians with  $q > 1$  are contained in the set of those with  $q < 1$ .

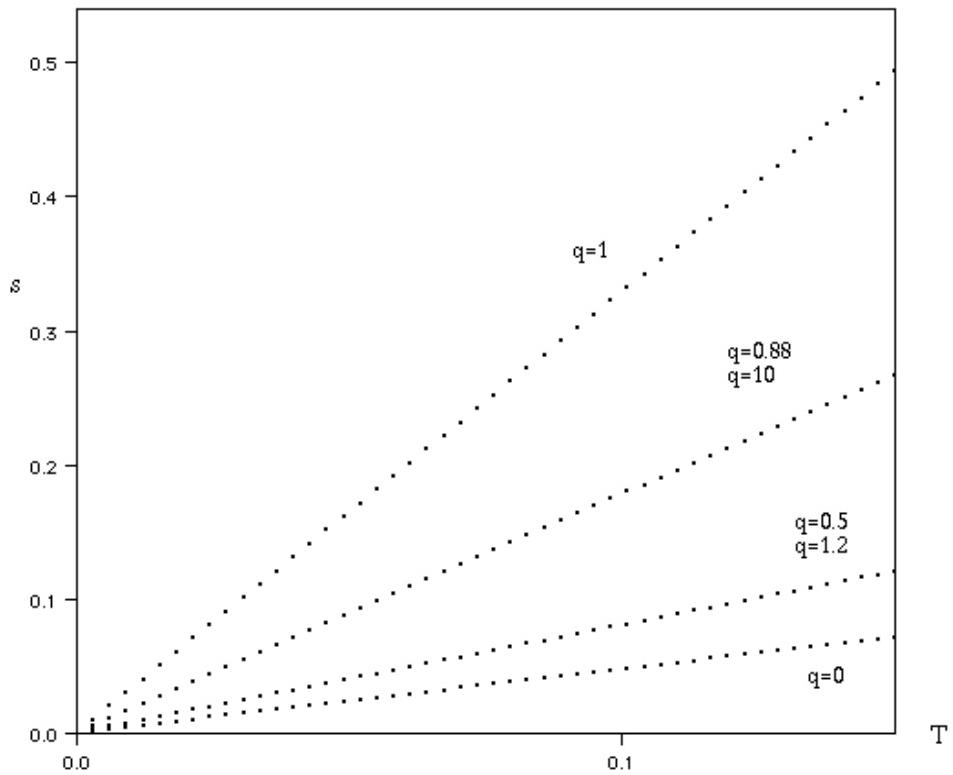


Figure 1: The function  $s$  as defined in the text for low temperatures and several values of  $q$

### High temperature and low density gas

In this section we calculate several thermodynamic functions for the case of weak degeneracy  $z = e^{\beta\mu} \ll 1$ , and study the role played by the parameter  $q$  in the equation of state. Our starting point is the grand partition function  $\mathcal{Z}$

$$\ln \mathcal{Z} = \frac{4\pi V}{(2\pi\hbar)^3} \int_0^\infty p^2 \ln \left( 1 + 2e^{-\beta(\varepsilon-\mu)} + e^{-\beta(\varepsilon(q^{-2}+1)-2\mu)} \right) dp, \quad (25)$$

such that expanding the integrand and keeping the first three terms gives

$$\ln \mathcal{Z} = \frac{\lambda\sqrt{\pi}}{\beta^{3/2}} \left[ \frac{z}{2} - \alpha(q) \frac{z^2}{2} + \gamma(q) \frac{z^3}{3!} + \dots \right], \quad (26)$$

where the functions  $\alpha(q)$  and  $\gamma(q)$  are

$$\begin{aligned} \alpha(q) &= \frac{1}{2^{3/2}} - \frac{1}{2(q^{-2}+1)^{3/2}} \\ \gamma(q) &= \frac{4}{3^{3/2}} - \frac{3}{(q^{-2}+2)^{3/2}}. \end{aligned}$$

Once we calculate the average number of particles  $\langle M \rangle = \frac{1}{\beta} \left( \frac{\partial \ln \mathcal{Z}}{\partial \mu} \right)_{T,V}$  we can write the fugacity  $z$  in terms of  $\langle M \rangle$ , and Equation (26) becomes

$$\ln \mathcal{Z} = \langle M \rangle \left[ 1 + \frac{2\alpha(q)\langle M \rangle}{\lambda\sqrt{\pi}} \beta^{3/2} - \frac{\langle M \rangle^2}{\lambda^2\pi} \beta^3 \Lambda + \dots \right], \quad (27)$$

where  $\Lambda = \frac{8\gamma(q)}{3} + 16\alpha^2(q)$ .

From this equation we can obtain the internal energy  $U$ , the heat capacity  $C_v$  and the entropy  $S = \frac{U-\mu\langle M \rangle}{T} + k \ln \mathcal{Z}$  as functions of  $\langle M \rangle$ . The corresponding equations are

$$U = \frac{3\langle M \rangle}{2\beta} + \frac{3\langle M \rangle^2 \beta^{1/2} \alpha(q)}{\sqrt{\pi}\lambda} - \frac{3\langle M \rangle^3 \beta^2}{2\pi\lambda^2} \Lambda + \dots, \quad (28)$$

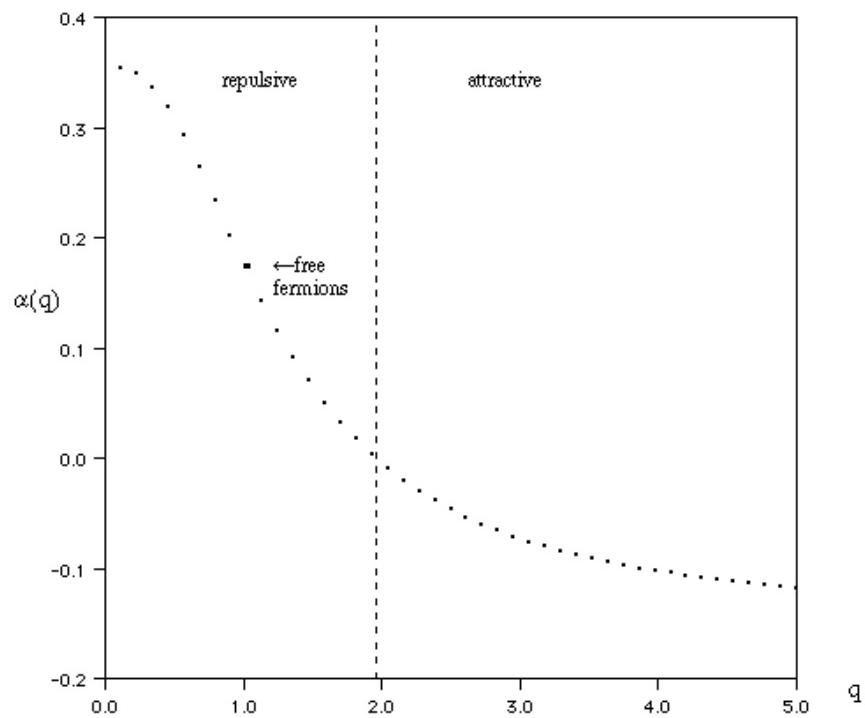


Figure 2: The coefficient  $\alpha(q)$  for the interval  $0 \leq q \leq 5$ . The line at  $q = 1.96$  divides the region between  $\alpha(q) > 0$  and  $\alpha(q) < 0$  which corresponds to fermionic and boson-like behavior respectively

$$C_v = \frac{3\langle M \rangle k}{2} \left[ 1 - \frac{\langle M \rangle \alpha(q) \beta^{3/2}}{\lambda \sqrt{\pi}} + \frac{2\langle M \rangle^2 \beta^3}{\lambda^2 \pi} \Lambda + \dots \right], \quad (29)$$

$$S = \langle M \rangle k \left[ \frac{5}{2} - \ln \left( \frac{2\langle M \rangle \beta^{3/2}}{\lambda \sqrt{\pi}} \right) + \frac{\alpha(q) \langle M \rangle}{\lambda \sqrt{\pi}} \beta^{3/2} + \dots \right]. \quad (30)$$

The equation of state is given by the equation

$$pV = kT\langle M \rangle \left[ 1 + \frac{2\alpha(q) \beta^{3/2} \langle M \rangle}{\lambda \sqrt{\pi}} + \dots \right], \quad (31)$$

with the second virial coefficient  $B_2(q)$

$$B_2(q) = \frac{\alpha(q)}{2} \left( \frac{\hbar^2}{2mkT\pi} \right)^{3/2}. \quad (32)$$

The sign of the second virial coefficient clearly depends of the value of  $q$ , therefore the parameter  $q$  interpolates between fermion-like and boson-like behavior. Figure 2 shows a graph of the coefficient  $\alpha(q)$  as a function of  $q$ . The function  $\alpha(q)$  takes values in the interval  $2^{-5/2} \leq \alpha \leq 2^{-3/2}$  for  $0 \leq q \leq 1$ , vanishes at  $q = 1.96$  and gets its lowest value  $\alpha(q) = -0.15$  in the limit  $q \rightarrow \infty$ . One question one could address is whether performing the same calculation in two dimensions it would lead to a similar behavior, such that a possible connection between the anyonic and our second virial coefficient for the two dimensional system could be established. Repeating the same procedure for two spatial dimensions one finds that the second virial coefficient becomes the function

$$B_2(q) = \frac{\pi \hbar^2 \beta}{2m(q^2 + 1)}, \quad (33)$$

which is positive for all values of  $q$ , showing therefore that this system does not exhibit anyonic behavior in two dimensions.

In this article we studied the behavior of a  $SU_q(2)$  fermionic gas at low and high temperatures. We calculated several thermodynamic functions for

the two cases  $q > 1$  and  $q < 1$ . Our results point out that at a low  $T$  the entropy value for a gas with  $q \neq 1$  is lower than the one for the Fermi case, and become the lowest for  $q \rightarrow 0$ . In particular, for  $0.33 \leq q < 0.91$  systems with  $q > 1$  share the entropy function with systems with  $q < 1$ . We studied the behavior of this model at high temperatures and obtained the equation of state as a virial expansion. We found that the second virial coefficient  $B_2(q)$  has a dependence on the parameter  $q$  such that it vanishes at  $q = 1.96$  and becomes negative for  $q > 1.96$ . Thus, as the parameter  $q$  varies from zero to infinity, this simple quantum group fermionic model describes a large set of models that spans from repulsive systems,  $B_2(q) > 0$ , for low values of  $q$  to attractive ones for large values of  $q$ . The cases  $q = 1$  and  $q = 1.96$  certainly describe a free fermionic system (at all orders) and an ideal gas (up to the second virial coefficient) respectively. This kind of interpolation between boson-like and fermion-like behavior is well known in two dimensions for anyonic systems [14, 15]. We have shown that the simplest  $SU_q(2)$  fermionic system plays a similar role in three spatial dimensions.

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